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Partially-Corrected Euler Method for Solution of ODE's

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Our goal is to solve the ODE:

$$\frac{dy}{dt} = f(t, y(t)) \tag{1}$$

where we are to advance the independent variables y by marching forward in time t using a step size h. Here time levels are denoted by subscripts. The "modified Euler" method (*e.g.*, [1, 2]) uses a forward-Euler step as a predictor to yield a provisional estimate $\tilde{y}_{i+1}$ of the dependent variables at the advanced time level:

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)\ . \tag{2}$$

The corrector step uses the average of f at the starting point of the step and f at the provisional point, instead of simply f at the starting point:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_{i+1}, \tilde{y}_{i+1}) + f(t_i, y_i)\right] \tag{3}$$

This is a better approximation to the Taylor series expansion of the solution. It yields an error of $O(h^3)$ per step, and a cumulative error of $O[h^2(b-a)]$ in the approximation to $y(b)$ that is obtained by applying the method over the interval $[a, b]$. Thus the method is "second order accurate," at the cost of two evaluations of the function f per step.

In contrast, the "partially-corrected Euler" method requires only a single function evaluation per step; nonetheless, it is also second order accurate. On the initial timestep, its predictor step is identical to that of the modified Euler method; however, for all subsequent steps the function evaluation "left over" from the previous step is used for the predictor. Thus the partially-corrected Euler method is:

$$\tilde{y}_{i+1} = y_i + hf(t_i, \tilde{y}_i) \tag{4}$$

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_{i+1}, \tilde{y}_{i+1}) + f(t_i, \tilde{y}_i)\right] \tag{5}$$

so that only in the corrector step is there an evaluation of f.

The above formulation requires that y_{i+1} and $f(t_{i+1}, \tilde{y}_{i+1})$ be saved at the end of each step. It also requires that y_i, $\tilde{y}_{i+1}$, and $f(t_i, \tilde{y}_i)$ be saved in the middle of each step after the predictor advance. It is possible to reduce the requirement for additional mid-step storage by rewriting eq. 5 as:

$$y_{i+1} = \tilde{y}_{i+1} + \frac{h}{2}\left[f(t_{i+1}, \tilde{y}_{i+1}) - f(t_i, \tilde{y}_i)\right] \tag{6}$$

so that at mid-step it is necessary to save only $\tilde{y}_{i+1}$ and $f(t_i, \tilde{y}_i)$, and this can be done using the same arrays employed for the end-of-step storage.

It is important to note that predictor-corrector methods are not symplectic [4, 5], and so in general they are not preferred for computing particle orbits; for an extensive discussion see [5]. Nonetheless, they remain of broad general utility, and the reduced computational effort of the partially corrected Euler method may be appreciated.

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Addendum - energy of a particle moving in harmonic well

The Modified Euler algorithm for particle motion in a harmonic well is:

$$\tilde{x}_{i+1} = x_i + hv_i \tag{7}$$
$$\tilde{v}_{i+1} = v_i - hx_i \tag{8}$$
$$x_{i+1} = (1 - h^2/2)x_i + hv_i \tag{9}$$
$$v_{i+1} = -hx_i + (1 - h^2/2)v_i \ . \tag{10}$$

The total energy, defined as $(x^2 + v^2)/2$, is obtained by squaring each of the latter two expressions:

$$x_{i+1}^2 = (1 - h^2/2)^2 x_i^2 + 2h(1 - h^2/2)x_i v_i + h^2 v_i^2 \tag{11}$$
$$v_{i+1}^2 = (1 - h^2/2)^2 v_i^2 - 2h(1 - h^2/2)x_i v_i + h^2 x_i^2 \tag{12}$$

and then taking their sum and dividing by 2:

$$(x_{i+1}^2 + v_{i+1}^2)/2 = \left[(1 - h^2/2)^2 + h^2\right](x_i^2 + v_i^2)/2 \tag{13}$$
$$= \left[1 + h^4/4\right](x_i^2 + v_i^2)/2 \tag{14}$$

so that the energy grows by a factor of $(1 + h^4/4)$ per timestep.

We have yet to work out the corresponding expression for the partially-corrected Euler method. However, a simple numerical test confirms that it affords damping of magnitude comparable to that of the growth associated with modified Euler, and a similar phase error. The leapfrog scheme (which, in contrast to the above methods, is particular to second-order ODE's) affords an oscillating energy and a smaller phase error. Use of a special energy measure, $(x_i^2 + v_{i-1/2}v_{i+1/2})/2$, for the leapfrog scheme yields a constant energy slightly smaller than the initial energy computed conventionally [6]. In a typical test, the particle was launched at $(x, y) = (1, 0)$, and followed for ten nominal periods using 80 steps per nominal period. The true orbit is a circle; Figure 1 shows the particle's location in the (x, v) phase space on successive "laps" for the three schemes, in an expanded view near $(1, 0)$. Ideally, the particle would always land at the centers of the small circles in the figure, returning to $(1, 0)$ on each lap. Figure 2 shows the time histories of the total particle energy $(x^2 + v^2)/2$ for the three schemes, using the conventional energy measure.

References

[1] C. DeTar, http://www.physics.utah.edu/~detar/phycs6720/handouts/ode/ode/node2.html .

[2] P. Hut and J. Makino, "Moving Stars Around," http://www.artcompsci.org/kali/pub/msa/ch10.html#rdocsect64 .

[3] The author encountered this method long ago under the name "partially corrected Euler" (and did not personally invent it), but has been unable to locate any references whatsoever. Perhaps it is well known, under another name. In any event, the author would appreciate any information that may be available regarding its origins.

[4] H. Yoshida, "Recent Progress in the Theory and Application of Symplectic Integrators," *Celestial Mechanics and Dynamical Astronomy* **56**, pp. 27-43 (1993).

[5] P. Hut and J. Makino, "The Art of Computational Science," http://www.artcompsci.org/index.html . See especially: http://www.artcompsci.org/kali/development.html , volume 2.

[6] J.-L. Vay, private communication, September 2007.

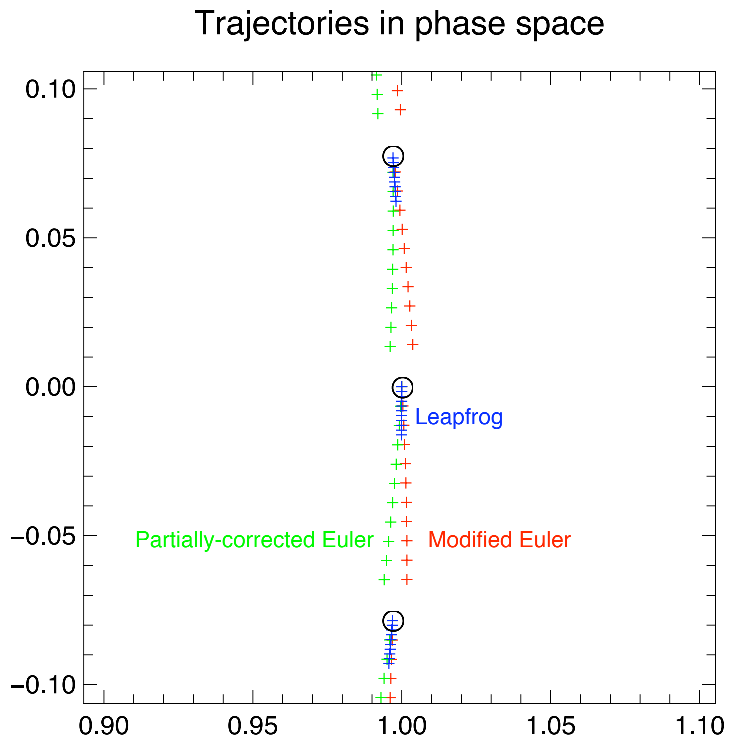

Figure 1: Particle locations in (x, v) phase space at successive transits of the orbit near the turning point at $+x$, for modified Euler (red), partially-corrected Euler (green), and leapfrog (blue) schemes.

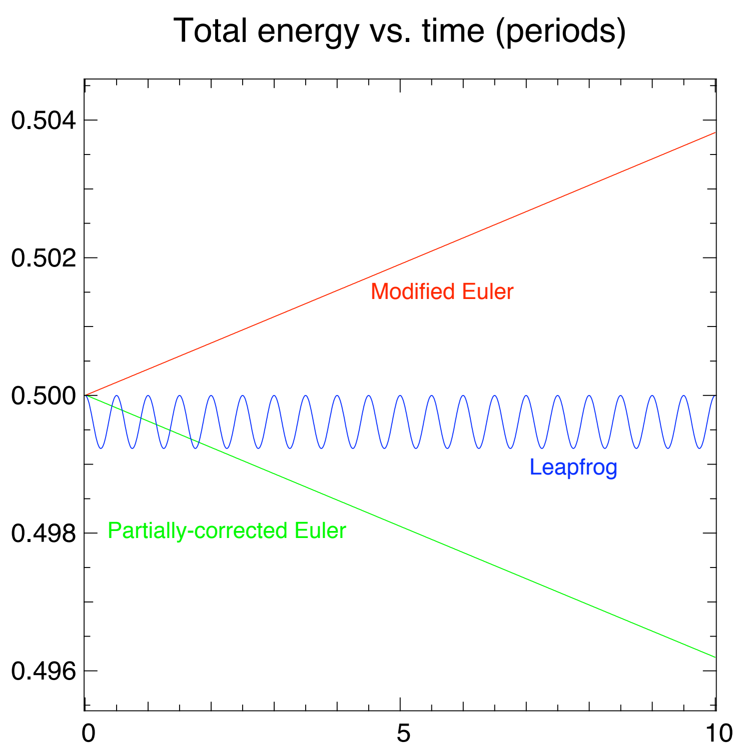

Figure 2: Evolution of total particle energy, for modified Euler (red), partially-corrected Euler (green), and leapfrog (blue) schemes.